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Classical Radiation from a Uniformly Accelerated Charge*

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The old and much-debated question, whether a charge in uniform acceleration radiates, is discussed in detail and its implications are pointed out. Many contradictory statements in the literature are analyzed and those answers which can be given on the basis of the standard classical Maxwell–Lorentz equations are presented. Although the questions that remain open are difficult and fundamental, some simple results can be proved: Contrary to claims in some standard sources (Pauli, von Laue), a charge in uniform acceleration *does* radiate. The radiation rate is finite, invariant, and constant in time in the instantaneous rest system. There is no contradiction of this fact with either the principle of conservation of energy or the principle of equivalence. Finally, the group of conformal transformations is found to be not physically meaningful.

1. WHAT IS THE PROBLEM?

Just fifty years ago, in 1909, Born published a paper (*1*) on the relativistic motion of a uniformly accelerated charge[1] (hyperbolic motion). In particular, he derived the electromagnetic fields associated with this motion. On the basis of these solutions of the Maxwell–Lorentz equations, Pauli then gave a simple argument (*2*), according to which such a charge does not produce a wave field and correspondingly cannot radiate. Von Laue too mentions that there is no radiation (*3*). This conclusion seems to have been confirmed by the discovery that the Maxwell equations are invariant under the group of conformal transformations (*4*), of which the Lorentz transformations are a subgroup. Since the conformal group contains in particular the transformation from rest to uniformly accelerated motion, it is concluded that such a motion cannot be associated with the (irreversible) emission of radiation. The defining equation for a conformal transformation yields the immediate result that the conventional form of the radiation reaction vanishes identically, and since it vanishes, it seems that the radiation must vanish as well.

Apparently, quite independent of Born's work, Schott also derived the fields

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[1] The precise definition of this term is given at the beginning of Section 2.

of a charge in hyperbolic motion (*5*) and concluded from his solutions that there is radiation (*6*). His work was discussed in further detail by Milner (*7*). Neither Schott nor Milner ever refers to Born's paper. Nor does Pauli mention Schott's conclusion that there is radiation.

More recently, Drukey (*8*) published a short note, presenting arguments in favor of radiation and apparently disproving previous arguments to the contrary. Finally, Bondi and Gold (*9*) have asserted that the Born solution did not treat the singularity of the potentials on the light cone correctly. They eliminated this difficulty by adding a δ-function to Born's solution, and they claimed that this δ-function provides proof for the radiation from a uniformly accelerated charge. They also point out that the fact that there is radiation leads to a contradiction with the principle of equivalence, because the emission of radiation would permit an observer to distinguish between the free fall of a charge in a gravitational field and its motion in field free surroundings. They resolve this contradiction by arguing that hyperbolic motion requires an *infinite* homogeneous gravitational field and that such a field does not exist in nature.

As the above brief sketch indicates, the problem of radiation from a uniformly accelerated charge is a very controversial one. Clearly, the difficulties do not lie in the computations involved, which are elementary, but rather in the understanding of the physical meaning implied in the equations, their solutions, and the conditions imposed on these solutions. Nor have many of the basic questions been satisfactorily answered, since the literature of the subject contains many discursive and qualitative, rather than rigorous and detailed arguments. Some of the unsettled questions are: Is Pauli's proof in error? If it is correct and if therefore uniformly accelerated charges do not radiate, where does the proof of the well-known radiation formula, found in the standard textbooks, break down? The standard formula implies that a charge will radiate whenever its acceleration does not vanish. On the other hand, if Pauli's proof is *in*correct, where do the arguments based on conformal invariance break down? Can radiation be emitted when the radiation reaction vanishes? How can energy be conserved if the radiation reaction is zero, and the work done by the external field equals the change of the kinetic energy of the particle? In connection with gravitation such questions arise as: "Do charged and neutral particles fall equally fast in a uniform gravitational field?" "Is there a contradiction between classical electromagnetic theory and the principle of equivalence in this case?"

We see that this problem involves basic concepts and principles both of classical relativistic electrodynamics and of general relativity.

The purpose of the present paper is to answer the questions raised above within the framework of generally accepted theory. Thus, we plan to prove such statements as the following:

"If the Maxwell–Lorentz equations are taken to be valid, and we consider retarded potentials only, and if radiation is defined in the usual Lorentz invariant manner, a uniformly accelerated charge radiates at a constant nonvanishing rate."

Or: "If one accepts the equations of motion based on the Abraham four-vector or on Dirac's classical electrodynamics, the radiation reaction vanishes, but energy is still conserved."

We furthermore point out the theoretical framework necessary in order to answer the various questions raised in the literature, and we show that within one of the possible known theories a consistent treatment of the problem can always be given and all apparent contradictions can be removed. Thereby we do not necessarily mean to advocate any single one of the theories of the classical electron. At the same time we do not of course question the validity of Maxwell's equations in the classical domain.

Our emphasis will be on rigorous calculations based on standard classical electrodynamics and Lorentz invariance, rather than on discursive arguments. In the following sections we consider first the consequences of the Maxwell–Lorentz equations (Sections 2 and 3), then the relativistic equations of motion for a charge (Section 4), and finally the problems raised by the theory of gravitation (Section 5). In the last section we summarize our conclusions and suggest directions for future work.

2. POTENTIALS AND FIELD STRENGTHS

A particle is said to be in uniformly accelerated motion when it experiences constant acceleration $[a'^{\mu} = (0,\mathbf{a}'),\ \mathbf{a}' = \text{const.}]$ in its instantaneous rest system $S'[v'^{\mu} = (1,0,0,0)]$.[2] Such motion can be produced by a constant, homogeneous gravitational field or, when the particle is charged, by a constant, homogeneous electric field. (See Eq. (4.4) below.)

The assumption that such a motion is possible for a finite period of time implies that the field equations (i. e., the Maxwell–Lorentz equations) and the equation of motion of the charged particle, when solved simultaneously, have this motion as a solution. Since the simultaneous solution of these equations is difficult, we assume the motion in question to be a possible solution and verify this assumption by studying the field equations and the equations of motion separately.

Let S be an inertial system in which the motion of a particle is described by $\mathbf{r}(t)$ and $\mathbf{v}(t) = d\mathbf{r}/dt \equiv \dot{\mathbf{r}}$. If this particle is to be uniformly accelerated according to the above definition, then

$$(1 - v^2)\ddot{\mathbf{v}} + 3\dot{\mathbf{v}}(\dot{\mathbf{v}}\cdot\mathbf{v}) = 0 \tag{2.1}$$

[2] We use the metric $g^{\mu\nu} = (-1,1,1,1)$ for $\mu = \nu = 0,1,2,3$; and $g^{\mu\nu} = 0$ for $\mu \neq \nu$. We put $c = 1$ but otherwise use Gaussian units, since we shall be concerned primarily with fields, not with field equations.

is equivalent to this definition (*4*). As was shown by Hill, the solution of this equation is exactly the four-dimensional group of conformal transformations in space-time. Let $\mathbf{a}'$ be in the direction of the z-axis, $\hat{\mathbf{z}}$, and choose S such that $+\mathbf{v}$ or $-\mathbf{v}$ will always be in the direction $\hat{\mathbf{z}}$. Then the only nontrivial case of uniform acceleration ($\mathbf{a}' \neq 0$) is that of hyperbolic motion. Without loss of generality we can assume that at $t = 0$ the particle is at $z = \alpha > 0$. Then

$$z = \sqrt{\alpha^2 + t^2}, \qquad \mathbf{v} = \frac{t}{\sqrt{\alpha^2 + t^2}}\,\hat{\mathbf{z}}, \tag{2.2}$$

and one easily verifies that $|\mathbf{a}'| = 1/\alpha$. (See Eq. (4.4) below.)

Given the motion (2.2) of a charge e, the electromagnetic fields produced by it are determined by the Maxwell–Lorentz equations. Their formal solutions are the Liénard–Wiechert potentials

$$A_\mu^P = e\frac{v_\mu^Q}{R^\nu v_\nu^Q}; \qquad R^\mu = (t - t_Q, \mathbf{r} - \mathbf{r}_Q), \tag{2.3}$$

where $P{:}(t,\mathbf{r})$ refers to the observation event (field point), $Q{:}(t_Q, \mathbf{r}_Q)$ to the emission event (source point), and v_μ^Q is the "retarded" four velocity. The observation time is related to the emission time by the *causality condition*

$$t - t_Q = R \equiv |\mathbf{r} - \mathbf{r}_Q| > 0, \tag{2.4}$$

which is implied in (2.3) and which assures that (2.3) refers to the *retarded* potentials.

Because of the equation of motion for the particle, $\mathbf{r}_Q$ is a function of t_Q, and because of the causality condition (2.4), t_Q is an implicit function of P. One can therefore express A_μ^P explicitly as a function of r and t only. (In the following we shall omit the superscript P.) The motion (2.2) then gives

$$\begin{aligned} \phi^B(\rho,z,t) &= e\,\frac{z(\rho^2 + z^2 + \alpha^2 - t^2) - \xi t}{\xi(z^2 - t^2)}, \\ A_\rho^B = A_\phi^B = 0; \qquad A_z^B(\rho,z,t) &= e\,\frac{t(\rho^2 + z^2 + \alpha^2 - t^2) - \xi z}{\xi(z^2 - t^2)}, \\ \xi &= +[(\alpha^2 + t^2 - \rho^2 - z^2)^2 + 4\alpha^2\rho^2]^{1/2}, \end{aligned} \tag{2.5}$$

where we have used cylindrical coordinates ρ, ϕ, z. The field strengths follow by differentiation:

$$\begin{aligned} E_\phi^B &= 0, \\ E_z^B &= -e\cdot 4\alpha^2(\alpha^2 + t^2 + \rho^2 - z^2)/\xi^3, \\ E_\rho^B &= e\cdot 8\alpha^2\rho z/\xi^3, \\ H_\rho^B &= H_z^B = 0, \\ H_\phi^B &= e\cdot 8\alpha^2\rho t/\xi^3. \end{aligned} \tag{2.6}$$

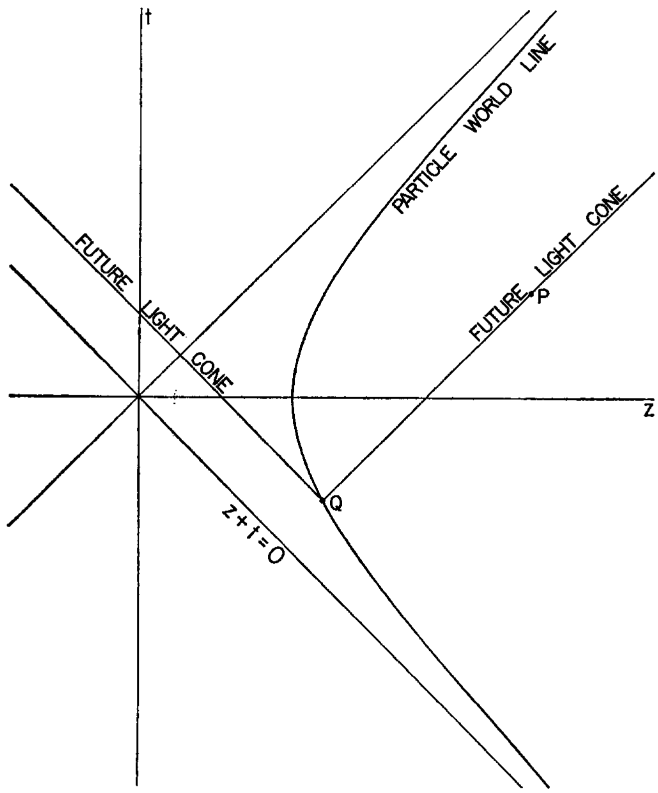

FIG. 1. Forward light cone from a source point Q during hyperbolic motion

The causality condition (2.4) is always appended to the Liénard–Wiechert potentials (2.3). However, it is lost in the Eqs. (2.5). The reason for this is that R has the form:

$$R = \tfrac{1}{2}[z\xi - t(\alpha^2 + t^2 + \rho^2 - z^2)]/(z^2 - t^2),$$

and there is no restriction on z, ρ, t which would prevent R from becoming negative. Since $R > 0$ ($R < 0$) is equivalent to $z + t > 0$ ($z + t < 0$), we must add to the solutions (2.5) and (2.6) the causality condition that these solutions are restricted to $z + t > 0$. That the causality condition is equivalent to the statement $z + t > 0$ can be seen from Fig. 1. We have plotted the geodesic of the particle in this diagram. The intersection of a typical future light cone with the z–t plane is also illustrated. It is clear from Fig. 1 that the family of all future light cones must lie in the region $z + t > 0$.

This restriction to $z + t > 0$ was apparently overlooked by Born, or at least was explicitly stated first by Schott. The results (2.5) and (2.6) without the restriction are thus referred to as the *Born solution* and have been labeled accordingly. The *Schott solution* is identical with the Born solution except for the restriction to the space-time region $z + t > 0$.

The Born solution therefore does not describe the fields of our problem. Rather, it describes the retarded fields of a particle of charge e performing the motion $z = \sqrt{\alpha^2 + t^2}$ (as in our problem) for $z + t > 0$, together with the advanced fields of a particle of charge $-e$ performing the motion $z = -\sqrt{\alpha^2 + t^2}$ for $z + t < 0$. This point was discussed by Milner (*7*).

It should be noted that the causality condition restricts our solutions to the open region $z + t > 0$ rather than the closed region $z + t \geqq 0$. (Note that fields on $z + t = 0$ would have to have been emitted at a time $t = -\infty$ when the charge was moving on the light cone at $z = +\infty$. A particle of finite mass can only approach the light cone but cannot reach it. Another way of seeing this exclusion of $z + t = 0$ is to note that, although the Maxwell–Lorentz equations for $F^{\mu\nu}$ as given by (2.6) are satisfied in the *open* region $z + t > 0$, they are not satisfied for $z + t = 0$ if (2.6) is assumed to be valid in the *closed* region $z + t \geq 0$ (with $F^{\mu\nu} = 0$ in $z + t < 0$).

Bondi and Gold (*9*) required that the field equations be satisfied everywhere, and modified the Born solutions by using a limiting process. Their results can be obtained much more easily by adding an undetermined function to the Born solution such that (2.6) holds for $z + t > 0$; $F^{\mu\nu} = 0$ for $z + t < 0$, and the modified solution satisfies Maxwell's equations also for $z + t = 0$. One then finds that such a modification is unique and yields

$$\begin{aligned} E_\phi &= 0, \qquad H_\rho = H_z = 0, \\ E_\rho &= E_\rho^B \theta(z + t) + \frac{2e\rho}{\rho^2 + \alpha^2} \delta(z + t), \\ E_z &= E_z^B \, \theta(z + t), \\ H_\phi &= H_\phi^B \, \theta(z + t) - \frac{2e\rho}{\rho^2 + \alpha^2} \delta(z + t). \end{aligned} \tag{2.7}$$

The function $\theta(x)$ is defined by

$$\theta(x) = \begin{cases} 1 \text{ for } x^0 > 0 \\ \tfrac{1}{2} \text{ for } x^0 = 0 \\ 0 \text{ for } x^0 < 0, \end{cases}$$

and δ is the Dirac delta-function. The result (2.7) agrees with that of Bondi and Gold, except for a sign in E_ρ (which seems to be a misprint) and the absence of the θ-function in their work (which modifies the fields at $z + t = 0$.) The fields (2.7) satisfy the Maxwell–Lorentz equations everywhere and contain the causality condition (2.4) implicitly.

The Bondi–Gold field strengths (2.7) can be derived from potentials which are suitable modifications of (2.5), viz.,

$$\phi = \phi^B\theta(z+t) - e\ln\left(\frac{r^2+\alpha^2}{\alpha^2}\right)\delta(z+t),$$
$$A_\rho = A_\phi = 0, A_z = A_z^B\theta(z+t) + e\ln\left(\frac{r^2+\alpha^2}{\alpha^2}\right)\delta(z+t), \tag{2.8}$$

and which differ from the Liénard–Wiechert potentials. The Liénard–Wiechert potentials are not valid in our case at $z + t = 0$, because their derivation assumes that the source is *not* at infinity. On the other hand, as mentioned above, the fields (2.7) and potentials (2.8) for $z + t = 0$ arise from the time $t = -\infty$ when the charge was at $z = +\infty$ (see Fig. 1).

Actually, a hyperbolic motion should be regarded as asymptotic in the sense that the times $t_Q = \pm\infty$ can only be approached but never reached. This corresponds to restricting the domain of validity of the fields (2.6) to the region $z + t > 0$. In this way, one avoids the unphysical event, $z_Q = +\infty$, $t_Q = -\infty$, where a particle of finite mass has *exactly* the velocity of light, and is being decelerated. Admittedly, this is possible within the framework of Maxwell's equations, but in view of the fact that eventually the motion is also to satisfy certain relativistic equations of motion, this event must be excluded. In any case, we shall see that the question of whether there is radiation *during* the hyperbolic motion has nothing to do with whether (2.6) or (2.7) is adopted. For these reasons we feel that the modification suggested by Bondi and Gold is not physically meaningful, and any conclusions drawn from it do not necessarily apply to emission times $t \neq -\infty$.

What are the fields seen by an observer in the inertial frame which is the instantaneous rest frame of the charge? We note that at time $t = 0$, $\mathbf{v} = 0$ in S so that S *is* the rest system of the charge at a particular instant. The fields present at that instant, however, are the fields produced *before* the charge reached the position $z = \alpha$ and are therefore much more complicated than a pure Coulomb field. Let S' be another inertial system which moves along the z-axis with velocity $\mathbf{v}$ ($v < 1$) relative to S. Since hyperbolic motion involves all velocities $\mathbf{v}$ ($v < 1$), there will be an instant t in S for which S' is the rest system. A Lorentz transformation to that system shows that this instant occurs at $t' = 0$ and that then the charge is at $z' = \alpha$. One easily verifies that the complete hyperbolic motion remains invariant. In particular, the fields in the frame S' satisfy the relation

$$F'_{\mu\nu}(\rho'\,\phi'\,z'\,t') = F_{\mu\nu}(\rho'\,\phi'\,z'\,t'), \tag{2.9}$$

i.e., they are form invariant in any inertial frame.

Now the Born solution is obtained from the Maxwell–Lorentz equations *without* the restriction of the causality condition. Therefore, since these equations are invariant under conformal transformations, the Born solution must also be so invariant. The form invariance of $F^{\mu\nu}$ could therefore have been anticipated. However, the Born solution is a sum of retarded (for $z + t > 0$) and advanced (for $z + t < 0$) solutions for two charges, as explained above, and *only this particular combination* remains invariant under conformal transformations. The retarded fields alone (Schott solution) remain invariant only under Lorentz transformations. The causality condition therefore restricts the symmetry of the fields to this proper subgroup of the conformal transformations. This fact seems to indicate that—at least within the framework of special relativity—conformal transformations do not play a fundamental role in physics. (A more detailed discussion of this point is given in the Appendix.)

3. DOES A UNIFORMLY ACCELERATED CHARGE RADIATE?

In order to answer this question it is necessary first to define what we mean by radiation. If one thinks in terms of quantum mechanics, a photon, when emitted, can be registered by a counter which is located at a large distance from the source. Clearly, such an effect can be seen by *every* observer. For this reason it is natural to try to define radiation in classical mechanics in a way that is invariant at least for all inertial observers. Furthermore, this definition must be such that it reduces to the conventional definition in terms of the Poynting vector in the rest system of the source.

Let

$$4\pi T^{\mu\nu} = -F^{\mu\lambda}F^{\nu}{}_{\lambda} + \tfrac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$$

be the energy-momentum tensor of the radiation field, such that $T^{k0} = -S_k$ where $\mathbf{S} = (S_1, S_2, S_3)$ is the Poynting vector.[3] Let $v_Q{}^{\mu}$ as before be the velocity four-vector of the source of radiation at the instant when radiation is emitted (retarded velocity). Let n^{μ} be a space-like unit vector (orthogonal to $v_Q{}^{\mu}$). Then we adopt the following definition (*10*) for the energy flux density emitted at the event Q in a direction determined by n^{μ}:

$$I = T^{\mu\nu}v_{\mu}{}^{Q}\, n_{\nu}\,. \tag{3.1}$$

In the inertial system S_Q where $n^{\mu} = (0,\hat{\mathbf{n}})$ and $v_Q{}^{\mu} = (1,0,0,0)$ (i.e., S_Q is the rest system at the instant of emission),

$$I = \mathbf{S}\cdot\hat{\mathbf{n}}. \tag{3.2}$$

At each instant, the space part of the light cone is a light-sphere whose radius R will be large for large times and on whose two-dimensional surface element $d^2\sigma$ (with normal $\hat{\mathbf{n}}$) I is given by (3.2).

[3] Poynting's theorem, i. e., the fourth component of the conservation law $\partial_{\mu}T^{\mu\nu} = 0$ holds not only for the Schott solution, but—at least formally—also for the Bondi–Gold modification (2.7), although its verification involves squares of δ-functions. Poynting's theorem therefore does not provide an argument against the Bondi–Gold fields.

The total rate of radiation energy emitted at time t_Q is found by integrating invariantly over the surface of the light sphere in the limit of infinite $R = t - t_Q$ for a fixed emission time t_Q

$$\mathcal{R} = \int T^{\mu\nu} v_\mu^Q \, n_\nu \, d^2\sigma \qquad (\text{limit } R \to \infty, \text{ fixed } t_Q). \tag{3.3}$$

From (3.2), we have in the inertial frame S_Q

$$\mathcal{R} = \int \mathbf{S} \cdot \hat{\mathbf{n}} \, R^2 \, d\Omega \qquad (\text{limit } R \to \infty, \text{ fixed } t_Q). \tag{3.4}$$

If this limit is finite we say that there is radiation. Otherwise, i. e., when $\mathcal{R} \to 0$ as $R \to \infty$, no radiation is emitted.

Two remarks should be made in connection with (3.4). One remark refers to the interesting fact that (3.4) is valid also in any other inertial system, provided that a factor dt/dt_Q is inserted in the integral. This factor is to be computed from the causality condition (2.4) and has the physical meaning of making $\mathcal{R}$ an energy rate, dW/dt_Q, relative to the source time t_Q. It is not obvious that dW/dt_Q is a constant equal to $\mathcal{R}$ for hyperbolic motion, since dW is the time-component of a four-vector. However, the calculation below will prove this fact.

The other remark concerns the important point that R is defined as a limit for fixed t_Q, i. e., both R and t must approach ∞ so that the causality condition (2.4) continues to hold. With this end in mind, t is first eliminated by the use of (2.4) and then the limit $R \to \infty$ is taken.

It is worth emphasizing that $\mathcal{R}$ in (3.3) is an invariant despite the fact that the integration extends over a *two*-dimensional spherical surface. The reason is that this spherical surface is the light sphere, which is an invariant. Note, however, that $\mathcal{R}$ is not necessarily a constant, but a function of the source point Q, i.e., of the proper time, τ_Q. As we shall see presently, in hyperbolic motion $\mathcal{R}$ actually *is* a constant.

The radiation rate $\mathcal{R}$ can be computed quite generally for any given source point Q of given $v^\mu(\tau)$, using the corresponding Liénard–Wiechert potentials. One finds (*10*)

$$\mathcal{R} = \tfrac{2}{3} e^2 a_\mu a^\mu, \tag{3.5}$$

where $a_Q^\mu = dv_Q^\mu/d\tau$ is the (retarded) acceleration four-vector of the source. Equation (3.5) is clearly an invariant.

In our particular case it is of some interest to carry out this calculation explicitly. The fields (2.6) yield

$$\mathbf{S} = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{H}) = \frac{1}{4\pi} H_\phi (E_\rho \hat{\mathbf{z}} - E_z \hat{\boldsymbol{\rho}}),$$

$$\mathbf{n} = \frac{\mathbf{R}}{R}; \qquad \mathbf{R} = \rho\hat{\mathbf{\rho}} + (z - \sqrt{\alpha^2 + t_Q^2})\,\hat{\mathbf{z}};$$

$$\frac{dt}{dt_Q} = 1 + \frac{dR}{dt_Q} = 1 - \mathbf{v}_Q \cdot \hat{\mathbf{z}} \cos\theta = 1 - \mathbf{v}_Q \cdot \hat{\mathbf{n}}; \qquad \cos\theta = \hat{\mathbf{z}} \cdot \hat{\mathbf{n}}.$$

After substitution, one finds

$$I = \frac{e^2\alpha^4}{4\pi} \cdot \frac{\sin^2\theta/R^2}{(\sqrt{\alpha^2 + t_Q^2} - t_Q \cos\theta)^6}. \tag{3.6}$$

Integration gives

$$\mathcal{R} = \frac{2}{3}\, e^2 \dot{\mathbf{v}}_Q^{\,2} \frac{1}{[1 - (\mathbf{v}_Q \cdot \hat{\mathbf{z}})^2]^3}. \tag{3.7}$$

Equations (3.6) and (3.7) are just the formulas which appear in standard textbooks on electrodynamics (*11*).

Using (2.2),

$$\dot{v}_Q = \frac{\alpha^2}{(\alpha^2 + t_Q^2)^{3/2}}, \qquad \mathbf{v}_Q \cdot \hat{\mathbf{z}} = \frac{t_Q}{\sqrt{\alpha^2 + t_Q^2}},$$

one finds

$$\mathcal{R} = \frac{2}{3}\frac{e^2}{\alpha^2}, \tag{3.8}$$

which, of course, is (3.5) for our special case. Thus, $\mathcal{R}$ is independent of Q and always has the same value as in the rest system.

We learn from this discussion that a charge in uniformly accelerated motion emits radiation at a constant rate (3.8) (in terms of retarded time) and with a radiation pattern given by (3.6).

Several questions can now be answered. First, it is clear that, according to our definition of radiation, every inertial observer will see exactly the same radiation intensity and same radiation rate. Radiation cannot be transformed away by a Lorentz transformation.

Secondly, it is clear that the radiation rate $\mathcal{R}$ can be computed for any arbitrary event on the hyperbolic world line. Consequently, the fields on $z + t = 0$ never enter this consideration. This means, in particular, that the result (3.8), $\mathcal{R} \neq 0$ follows from the *Born solution* in the region $z + t > 0$ *irrespective* of the value of this solution in the region $z + t \leq 0$. According to Bondi and Gold (*9*), radiation is emitted at $t = -\infty$ (producing the fields on $z + t = 0$); but nothing is proven concerning radiation emission at any other time. Our results prove that radiation fields are produced at every instant throughout the motion. Further, as we have already discussed, there is no physical meaning in the fields on $z + t = 0$ or in hyperbolic motion at $t = -\infty$.

Thirdly, it has now been shown by explicit calculation that Pauli's argument cannot be valid. Indeed, Pauli argues that since $\mathbf{H} = 0$ at $t = 0$, "hyperbolic motion constitutes a special case, for which there is no formation of a wave zone nor any corresponding radiation (*2*)." We note, however, that in this statement a limit to large distances ($R \to \infty$) is implied at a fixed time ($t = 0$). This limit is not in accordance with the definition of radiation. Following this definition, Eq. (3.3), the limit $R \to \infty$ is to be taken for a fixed source point (t_Q fixed), implying a limit $t \to \infty$ as well. It can easily be seen that the two limiting procedures do not give the same result. Pauli's limit of $\mathbf{S}R^2$ vanishes for *any* fixed time t; for the time $t = 0$, where $\mathbf{S} = 0$, it is simply a special case where this limit vanishes trivially. On the other hand, the limit of $\mathbf{S}R^2$ does *not* vanish when t_Q is held fixed. We conclude that the fact, that $\mathbf{H} = 0$ at $t = 0$, is unusual for accelerated motion and of some interest, but it has nothing to do with the presence or absence of radiation.

4. HOW CAN ENERGY BE CONSERVED?

The question of energy conservation involves the kinetic energy of the radiating charge. It is therefore linked in an essential way to the *equation of motion* of this charge. Everything said so far was independent of the particular equation of motion which an accelerated relativistic charged particle satisfies.

At this point, however, we are looking in vain for a generally accepted equation which, like Maxwell's equations, has been confirmed over and over again. In fact, only two different approaches to the problem have been studied in detail, both of which lead to the same equation of motion. Neither approach is very satisfactory. One is the expansion method of a finite sized electron in which the first two terms lead to the electromagnetic self-energy and the Abraham radiation reaction four vector. The other follows from Dirac's classical electrodynamics (*12*). Both lead to the equation of motion:

$$m \frac{dv^\mu}{d\tau} = F^\mu_{\text{ext}} + \Gamma^\mu; \tag{4.1}$$

$$\Gamma^\mu = \frac{2}{3} e^2 \left(\frac{d^2 v^\mu}{d\tau^2} - v^\mu \frac{dv_\lambda}{d\tau} \frac{dv^\lambda}{d\tau} \right) \tag{4.2}$$

If the externally applied force is electromagnetic, then

$$F^\mu_{\text{ext}} = eF^{\mu\nu} v_\nu \tag{4.3}$$

is the Lorentz force.

The important point about Eq. (4.1) is that we do not need to accept it. But if we do not accept it we cannot answer any question concerning energy conservation. On the other hand, *if we do accept it, we are also forced to accept all the immediate consequences of this equation.*

Let us therefore assume for the moment that (4.1) describes correctly the motion of a moving charge. Then we notice first that for uniformly accelerated motion $\Gamma^\mu = 0$.

[[*Proof:* The four-vector $v^\mu = dx^\mu/d\tau = (\gamma,0,0,\gamma v_z) = [(\sqrt{\alpha^2+t^2}/\alpha),0,0,t/\alpha)]$ upon differentiation leads to $a^\mu = dv^\mu/d\tau = \gamma dv^\mu/dt = [t/\alpha^2,0,0,(\sqrt{\alpha^2+t^2}/\alpha^2)]$. Thus,

$$a_\mu a^\mu = \frac{1}{\alpha^2}.$$

One more differentiation yields $d^2v^\mu/d\tau^2 = v^\mu/\alpha^2 = v^\mu a_\lambda a^\lambda$, which establishes $\Gamma^\mu = 0$ in the inertial system S. Therefore $\Gamma^\mu = 0$ in any system.]]

The physical meaning of $\Gamma^\mu = 0$ is that the force of radiation reaction vanishes. Thus, the equation of motion (4.1) simply becomes

$$m\frac{dv^\mu}{d\tau} = F^\mu_{\text{ext}}. \tag{4.1$'$}$$

The covariant force F^μ is related to the (Newtonian) three vector force $\mathbf{F}$ by

$$F^\mu = \gamma(\mathbf{F}\cdot\mathbf{v},\mathbf{F}),$$

since $F^\mu v_\mu = 0$. Therefore (4.1)$'$ yields

$$m\frac{d(\gamma\mathbf{v})}{dt} = \mathbf{F}_{\text{ext}},$$

and with (2.2)

$$\mathbf{F}_{\text{ext}} = m\frac{1}{\alpha}\hat{\mathbf{z}}, \tag{4.4}$$

which proves that the (Newtonian) force which produces uniform acceleration is a constant force. The acceleration it produces is

$$\mathbf{a} \equiv \frac{d(\gamma\mathbf{v})}{dt} = \frac{1}{\alpha}\hat{\mathbf{z}} = \mathbf{a}'.$$

Thus, the three-vector acceleration in S is constant and equal to the space-part of the acceleration four-vector in the instantaneous rest system S', since $a'^\mu = (0,\mathbf{a}')$.

This result permits us to *define* uniform acceleration in the inertial system S as that motion which is characterized by a constant Newtonian force $\mathbf{F}_{\text{ext}} = md(\gamma\mathbf{v})/dt$. This definition is completely equivalent to the previously given definition (see the beginning of Section 2) only for one-dimensional motion.

We now turn to the discussion of energy conservation which is based on the zero-component of Eq. (4.1)$'$ and which will establish that a vanishing radiation reaction does not necessarily imply "no radiation."

Let $T = m(\gamma - 1)$ be the kinetic energy of the particle. Then the zero-component of (4.1)′ is

$$\frac{dT}{d\tau} = F^0_{\text{ext}} = \frac{dW_{\text{ext}}}{d\tau}. \tag{4.5}$$

Therefore, the rate of work done by the external force is exactly equal to the increase in kinetic energy of the particle, as we know it to be the case, for example, for a neutral particle in a static gravitational field.

This conclusion seems to make the original question even more demanding: What supplies the energy which the particle radiates? Does (4.5) not prove that there cannot be radiation and that the usual notion is valid, viz., that no radiation reaction means no radiation?

Γ^0 is the rate of work done by the radiation reaction. Consider Γ^0 in detail. *In general* (i.e., when one does not assume that the acceleration is uniform),

$$\begin{aligned} \Gamma^0 &= \frac{2}{3}e^2\left(\frac{da^0}{d\tau} - \gamma a_\mu a^\mu\right); \\ \Gamma^0 &= \frac{2}{3}e^2\frac{da^0}{d\tau} - \gamma\mathcal{R}. \end{aligned} \tag{4.6}$$

$\mathcal{R}$ is the radiation rate (3.5) and is positive since a^μ is a space-like vector. We divide (4.1) by γ (note that $dt = \gamma d\tau$) and find

$$\frac{dT}{dt} - \left(\frac{dQ}{dt} - \mathcal{R}\right) = \frac{dW_{\text{ext}}}{dt}, \tag{4.7}$$

where

$$Q = \tfrac{2}{3}e^2a^0. \tag{4.8}$$

Thus, the equation of motion (4.1) leads to the energy conservation equation (4.7) for *any* motion. This equation has the following physical meaning: The rate of work done by the external force equals the rate of increase in kinetic energy *minus* the rate of work done by the radiation reaction. The latter consists of two parts, a reversible rate dQ/dt which can be positive or negative, and an irreversible rate $-\mathcal{R}$ which is never positive. The sum $dQ/dt - \mathcal{R}$ in general does not vanish. Since $\mathcal{R}$ is exactly the radiation rate, one sees that the energy lost in the form of radiation is entirely accounted for by part of the work done by the radiation reaction. On the other hand, the remaining part of this work also supplies an additional energy Q which may be positive or negative. Apparently, Q is to be interpreted as part of the internal energy of the charged particle. Like its kinetic energy it can be increased or decreased. Q has been named *acceleration energy* by Schott (*5*), since Q increases (decreases) when the velocity increases (decreases).

When the motion is periodic or bounded and one averages over sufficiently large times, the term dQ/dt vanishes. Thus, under these conditions, Q represents a fluctuating term and over long time intervals *all* the work of the radiation reaction goes into radiation. The same is true if one considers the energy balance between an initial and a final state of motion which are equal. It is from these cases that the idea: "no radiation reaction means no radiation" arose. Most problems of interest are of this type, like classical charged harmonic oscillators or betatron orbits. Hyperbolic motion is not of this type. In general, however, there is always an instantaneous acceleration energy as a consequence of the equations of motion (4.1).

In the case of uniform acceleration, Γ^0 is zero, i.e., the total work done by the radiation reaction force vanishes. Therefore

$$\mathcal{R} = \frac{dQ}{dt} > 0. \tag{4.9}$$

Eq. (4.9) can easily be verified directly. The internal energy of the electron, $m - Q$, therefore decreases while energy is being radiated.

This result seems to lead to a very unphysical picture: The accelerated electron decreases its "internal energy," transforming it into radiation. Does this mean that the rest mass of the electron decreases? An observer for whom the electron is momentarily at rest ($\mathbf{v} = 0$) will also find $a^0 = \gamma\mathbf{v}\cdot\mathbf{a} = 0$ and therefore $Q = 0$. Thus we obtain the comforting result that the change in internal energy of the particle does not affect its rest mass. Rather, the radiation energy is compensated by a decrease of that part of the field surrounding the charge, which does not escape to infinity (in the form of radiation) and which does not contribute to the (electromagnetic) mass of the particle (*13*).

No matter how one interprets the conclusions from (4.9), there is no contradiction with the principle of conservation of energy. If the emerging physical picture seems unsatisfactory, one can reject the equations of motion (4.1). This is a possible alternative, but then the question of energy conservation simply cannot be answered, because the equations of motion are unknown. It would be entirely inconsistent to accept the vanishing radiation reaction, but to reject the concept of acceleration energy, as long as both concepts derive from the same equation.

A third alternative, in view of the generality of (4.7), is to exclude all cases where the average $\langle dQ/dt\rangle_{\text{av}} \neq 0$. This means that a motion which leads to $\langle dQ/dt\rangle_{\text{av}} \neq 0$ is then *a priori* excluded as unphysical. In that case hyperbolic motion of a charged particle is not possible. Such a restriction seems rather artificial and *ad hoc*, and one may prefer to discard Eq. (4.1).

5. DOES RADIATION IN HYPERBOLIC MOTION CONTRADICT THE PRINCIPLE OF EQUIVALENCE?

From the last section we saw that if one accepts the equations of motion (4.1), one concludes that there is no radiation reaction, i.e., $\Gamma^\mu = 0$. The equation of motion for a charged particle in hyperbolic motion therefore differs in no way from that for a neutral particle, when both are accelerated by nonelectromagnetic forces. We conclude therefore that in a homogeneous constant gravitational field a neutral and a charged particle will follow the same trajectory with the same time dependence. In less precise but more picturesque language we could say, "If Galileo had dropped a neutron and a proton from the leaning tower of Pisa they would have fallen equally fast." According to (4.1), a charged and a neutral particle in a homogeneous gravitational field behave exactly alike, except for the emission of radiation from the charged particle.

This, however, is just the point where the principle of equivalence enters. A particle which is falling freely in a homogeneous gravitational field should appear to an observer who is falling with it, like a particle at rest in an inertial frame (field free surroundings). If we consider a *neutral* particle falling in a homogeneous gravitational field, this is indeed what happens. But when the particle is charged, the observer can establish the presence of a gravitational field by looking for radiation. If he observes radiation from the charge, he knows that he and the charge are falling in a gravitational field; if he observes no radiation, he knows that he and the particle are in a force free region of space.

The solution to this apparent difficulty is to be found by considering an actual measurement of radiation, using our definition in Section 3. Radiation is defined by the behavior of the fields in the limit of large distance from the source. Correspondingly, an observer who wants to detect radiation *cannot do so in the neighborhood* of the particle's geodesic. Rather, he must be at a large distance from it, where gravitational fields have different values. The principle of equivalence, however, is a *locally* valid principle, referring to the geodesic of the particle, whereas the discussion above shows that any observation of radiation is *not* a local observation. This point is implied in the related argument given by Bondi and Gold (*9*) who first discussed this difficulty.

Since all our considerations here refer to the framework of the special theory of relativity and, in particular, require the existence of inertial coordinate systems, we are forced to consider the principle of equivalence from a similar point of view. Whatever gravitational field we introduce for the purpose of comparing it with an inertial field, we must be sure to have a distribution of distant stars which define our inertial systems. This means in particular that any homogeneous gravitational field is necessarily of finite extent, imbedded as is were, in an inertial coordinate system. We remark parenthetically that an infinite homogeneous gravitational field does not exist within the framework of general relativity either. The nonexistence of infinite homogeneous gravitational fields assures that the observation of radiation (observer at large distance from the source) takes place outside the homogeneous part of the gravitational field.[4]

To clarify the point concerning the principle of equivalence further, consider a particle in a gravitational field. Whether or not it will radiate depends solely on its acceleration *relative to an inertial system* S. If the gravitational field is at rest in S, and if the particle is at rest in the gravitational field, there will be no radiation, because we have not only $\Gamma^\mu = 0$ but also $a^\mu = 0$, so that $\mathcal{R} = 0$ and $Q = 0$. On the other hand, if the particle falls freely in the gravitational field (which is at rest relative to S) then there will be radiation. Such a consideration shows that these two cases are physically *not* equivalent. However, since radiation cannot be observed locally, they are *equivalent locally*. For gravitation theory only local equivalence is important. Once again, we could paraphrase the above discussion in less precise language by saying: "An electron which falls freely in a uniform gravitational field embedded in an inertial frame will radiate, and one which sits at rest on a table in the same field will not radiate; and these two statements do not contradict the principle of equivalence."

6. WHAT REMAINS TO BE DONE?

We can briefly summarize our conclusions as follows: First, within the framework of the Maxwell–Lorentz equations, a charge in uniformly accelerated motion radiates at a constant and finite rate. Secondly, this radiation (like any radiation according to our definition) is Lorentz invariant but not conformally invariant. Thirdly, there is no radiation reaction, but there is energy conservation, *provided* one accepts some equation of motion (Abraham–Dirac); one is then also forced to accept the physical picture emerging from this equation. Otherwise, i.e., without accepting an equation of motion no answer concerning energy conservation or radiation reaction can be given. Finally, there is no contradiction with the principle of equivalence.

An experimental check of the validity of the equation of motion, or rather its consequences for the conservation of energy, is not feasible, since the ratio of radiated power to rate of gain of kinetic energy by a uniformly accelerated particle is negligible unless energy increments of the order of the rest energy of the particle are imparted to it in distances of the order of its classical radius (*14*). The question of direct detection of the radiation from a uniformly accelerated beam of charged particles (say, by looking with a detector at the part of a bounded trajectory where uniform acceleration takes place) is a more difficult one to discuss and lies beyond the scope of this paper.

The interesting questions at this point are the open questions. If one accepts

[4] Compare the discussion of similar ideas in Ref. *9*

the equation of motion (4.1) as correct, what is the physical meaning of the acceleration energy and the apparently arbitrarily large depletion of the charge's internal energy by radiation in the course of its motion? If one does not accept this equation of motion, what is the correct equation and the correct radiation reaction of a classical charged particle?

The above questions are restricted to classical physics and to the framework of the special theory of relativity. One can extend the scope of the problem in two directions, toward quantum electrodynamics and toward general relativity.

The coupled electromagnetic field and charged particle equations, though they lead by no means to a consistent theory in quantum electrodynamics, at least give correct answers in perturbation theory. It would therefore be interesting to study the problem of uniform acceleration in quantum electrodynamics, to see how the energy balance comes about.

The extension to general relativity involves the difficult questions of the relationship of metric and electrodynamics. What does radiation mean within this framework? Does a finite curvature of space modify the above problems in an essential way? In particular, could it lead to an exclusion of all motions with $dQ/dt \neq 0$ in a natural way? What does energy conservation mean in this framework?

Thus, we see that the questions that were answered all seem to have been very simple, but the questions that remain all seem very difficult.

We gratefully acknowledge stimulating discussions with our colleagues at Johns Hopkins and elsewhere.

APPENDIX[5]

The conformal transformation of interest is

$$x^{\mu'} = \lambda^2(x^\mu - s^2 a^\mu),$$
$$\lambda^2 = [1 - 2a^\mu x_\mu + s^2 a^\mu a_\mu]^{-1}, \tag{A1}$$

where $s^2 = x_\mu x^\mu$. This transformation satisfies $s'^2 = \lambda^2 s^2$ and has the nonrelativistic limit ($c \to \infty$, $\beta \to 0$, $\beta c^2 \to g/2$)

$$t' = t; \qquad \mathbf{r}' = \mathbf{r} + \tfrac{1}{2}\mathbf{g}t^2, \tag{A2}$$

where we defined $a^\mu = (0,\boldsymbol{\beta})$, $|\boldsymbol{\beta}| = \beta$, $|\mathbf{g}| = g$. Thus, the space-part of a^μ corresponds to one-half the nonrelativistic acceleration.

Assume $a^\mu = (0,0,0,1/\alpha)$ and consider a point in the z–t plane. Its transformation will be given by

[5] The following considerations originated in a discussion by one of the authors (F.R.) with Dr. F. Gürsey. It is a pleasure to thank Dr. Gürsey for his valuable comments.

$$z' = \frac{\alpha\left[t^2 + \left(\frac{\alpha}{2}\right)^2 - \left(z - \frac{\alpha}{2}\right)^2\right]}{(z - \alpha)^2 - t^2},$$
$$t' = \frac{\alpha^2 t}{(z - \alpha)^2 - t^2}. \tag{A3}$$

This transformation obviously breaks down for $t^2 = (z - \alpha)^2$. Furthermore, t and t' will have the same signs for $t^2 < (z - \alpha)^2$ and opposite signs for $t^2 > (z - \alpha)^2$. A particle trajectory in S between $t = -|z - \alpha|$ and $t = +|z - \alpha|$ will be mapped into an infinite trajectory in S', between $t' = -\infty$ and $t' = +\infty$.

As an example, consider a particle at rest in S at a position $z = \alpha/2$. Its world-line is shown in Fig. 2(a).

The transformation (A3) yields

$$z' = \alpha \frac{\left(\frac{\alpha}{2}\right)^2 + t^2}{\left(\frac{\alpha}{2}\right)^2 - t^2},$$
$$t' = \frac{\alpha^2 t}{\left(\frac{\alpha}{2}\right)^2 - t^2}, \tag{A4}$$

so that $z'^2 - t'^2 = \alpha^2$. The world-line in S' is therefore a hyperbola (Fig. 2b).

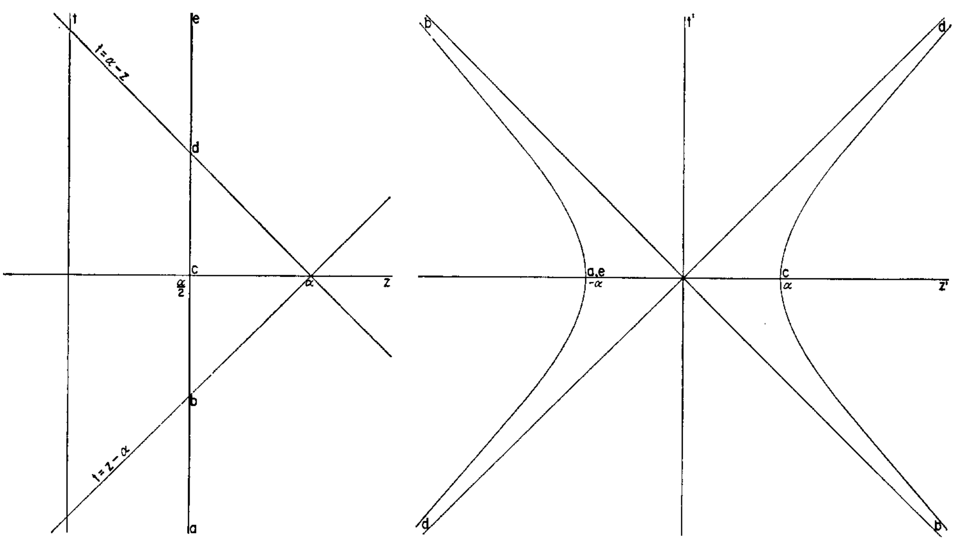

FIG. 2. Transformation of a particle at rest (a) to uniformly accelerated motion (b)

Closer inspection shows that the world-line a-b-c-d-e in S is transformed into *both* branches of the hyperbola with jumps at b and d as indicated. In particular the whole right-hand branch b-c-d in S' comes from the *finite* time interval b-c-d in S.

One concludes that the transformation (A1) cannot be regarded as a physically meaningful transformation of the world-line a–e into uniformly accelerated motion. Note, however, that in the nonrelativistic limit $\alpha \rightarrow \infty$, and the singular lines $t^2 = (z - \alpha)^2$ disappear.

The example also explains why a conformal transformation always leads to fields of *both* branches of the hyperbolic motion.

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